BORSUK–ULAM TYPE THEOREMS FOR METRIC SPACES

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Abstract. In this paper we study the problems of the following kind: For a pair of topological spaces X and Y find sufficient conditions that under every continuous map $f : X \to Y$ a pair of sufficiently distant points is mapped to a single point.

1. Introduction

In this paper we are going to give new proofs, using the recent ideas of M. Gromov, to the classical Borsuk–Ulam and Hopf theorems and their generalizations, and study some their consequences, as well as separate results, in the spirit of Urysohn width and Gromov waist.

Recall the famous Borsuk–Ulam theorem [3]:

Theorem 1.1 (K. Borsuk, S. Ulam, 1933)**.** *Under any continuous map $f : S^n \to \mathbb{R}^n$ some two opposite points are mapped to a single point.*

A deep generalization of this result is the Hopf theorem [7]:

Theorem 1.2 (H. Hopf, 1944)**.** *Let X be a compact Riemannian manifold of dimension n and $f : X \to \mathbb{R}^n$ be a continuous map. For any prescribed $\delta > 0$, there exists a pair $x, y \in X$ such that $f(x) = f(y)$ and the points x and y are connected by a geodesic of length δ.*

After these results, different definitions of *width* of a map were introduced by Kolmogorov, Alexandrov, and Urysohn; see the definitions in the book [14] (and the book [13] in English). Generally, the width is some guaranteed size (in some sense) of a preimage of a point $y \in Y$ under any continuous (or linear in some definitions) map $f : X \to Y$, see Section 6 for more precise definitions. Even for the simplest domain spaces like the Euclidean unit ball B^n and the round sphere (sphere with the standard Riemannian metric) S^n not all values of such widths are known.

In the papers [5, 12] another particular kind of width (called *waist* there)[1] was studied, based on the volume of the preimage of a point, and the waist of the round sphere was found to be the volume its equatorial subsphere of appropriate dimension. In Section 7

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[1]In fact "width" and "waist" correspond to the same Russian word "poperechnik".

we give some results on this kind of width/waist too, though our present technique is limited to the simple case of codimension 1 map.

Our presentation is greatly inspired by the results of [6], where the estimates for the size of the preimage of a point were proved using the technique of "contracting in the space of (co)cycles". One of the questions addressed in this paper is how this technique can be applied to the Borsuk–Ulam and Hopf theorems. Such an application turns out to be possible and these old theorems are generalized (see Theorems 2.3 and Theorem 4.1).

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2. A Borsuk–Ulam type theorem for metric spaces

We are going to utilize the ideas of M. Gromov [6] to give a coincidence theorem. Let us make a few definitions.

Definition 2.1. Let X be a topological space. Denote PX the *space of paths*, i.e. the continuous maps $c : [0, 1] \to X$. This space has a natural $\mathbb{Z}_2$-action by the change of parameter $t \mapsto 1 - t$, and a natural $\mathbb{Z}_2$-equivariant map

$$\pi : PX \to X \times X, \quad c \mapsto (c(0), c(1)).$$

Definition 2.2. Call a $\mathbb{Z}_2$-equivariant section s of the bundle $\pi : PX \to X \times X$ over an open neighborhood $\mathcal{D}(s)$ of the diagonal $\Delta(X) \subset X \times X$ a *short path map*, iff $s(x, x)$ is a constant path for any $x \in X$.

Such short path maps may be given by assigning a shortest path to a pair of points in a metric space. If X is a compact Riemannian manifold then such short path maps do exist.

Now we are ready to state:

Theorem 2.3. *Suppose X is a compact manifold of dimension n, Y is another manifold of dimension n, and $f : X \to Y$ is a continuous map of even degree. Then for any short path map $s : X \times X \to PX$ there exists a pair $(x, y) \notin \mathcal{D}(s)$ such that $f(x) = f(y)$.*

The classical Borsuk–Ulam theorem [3] follows from this theorem by considering $X = S^n$ and s to be the shortest path map in the standard metric. Theorem 1.2 (of Hopf) does not follow from this theorem because here we may only obtain an inequality on $\operatorname{dist}(x, y)$. An advantage of Theorem 2.3 is that the codomain Y may be arbitrary.

3. Space of cycles and the proof of Theorem 2.3

We start from explaining the main ideas underlying what Gromov calls "contraction in the space of cycles" [6] in a particular case. Denote $cl_0(X; \mathbb{F}_2)$ the space of 0-cycles mod 2 in X, that is the space of formal finite combinations $\sum_{x\in X} a_x x$ with $a_x \in \mathbb{F}_2$ with an appropriate topology.

A more tangible description of $cl_0(X; \mathbb{F}_2)$ is the union over $k \geq 0$ of spaces of unordered $2k$-tuples $B(X, 2k) \subset X^{\times 2k}/\Sigma_{2k}$. Informally, the topology in $cl_0(X; \mathbb{F}_2) = \bigcup_{k\geq 0} B(X, 2k)$ is such that when two points of a set $c \in B(X, 2k)$ tend to a single point then they "annihilate" giving a configuration in $B(X, 2k - 2)$ in an obvious way, and conversely a pair of points may be "created" from a single point giving a configuration in $B(X, 2k+2)$.

In the case when X is an n-dimensional manifold we define the *canonical class* ξ in $H^n(cl_0(X; \mathbb{F}_2); \mathbb{F}_2)$ as follows. Any n-dimensional homology class of $cl_0(X; \mathbb{F}_2)$ can be represented by a chain c, which is given by a map of an n-dimensional mod 2 pseudomanifold K to $cl_0(X; \mathbb{F}_2)$. Considering any element of $cl_0(X; \mathbb{F}_2)$ as a subset of X we may consider c as a set valued map from K to X. Its graph Γ_c is a subset of $K \times X$, which is again a

mod 2 pseudomanifold. Hence the degree mod 2 of the natural projection $\Gamma_c \to X$ is well defined. This degree will be the value $\xi(c)$ by definition. Another informal way to define ξ would be to count how many times a generic point $x_0 \in X$ participates in the 0-cycles from the chain c.

Now we return to the proof of the theorem. From the compactness considerations it is sufficient to prove the theorem for smooth generic maps f. In this case we may define the natural map

$$f^c : Y \to cl_0(X; \mathbb{F}_2),$$

which maps any $y \in Y$ to the $\mathbb{F}_2$-cycle

$$f^c(y) = \sum_{x \in f^{-1}(y)} c_x x,$$

where $c(x)$ is the mod 2 multiplicity of the map f at x. This map is well-defined because the degree of f is even by the hypothesis. The image of f^c represents an n-dimensional mod 2 homology class in $cl_0(X; \mathbb{F}_2)$ and by the definition of the fundamental class ξ it is obvious that $\xi(f^c(Y)) = 1$. Therefore the map f^c is homotopically nontrivial.

But we are going to deform the map f^c to the constant map by a homotopy h_t, using the short path map s. Put

$$h_t(y) = \sum_{x_1 \neq x_2 \in f^{-1}(y),\ c_{x_1}, c_{x_2} = 1} s(x_1, x_2)(t/2).$$

We have to check whether this map is continuous in y and t. If the preimage $f^{-1}(y)$ changes by "annihilating" a pair points or "creating" a pair of points, the components of $h_t(y)$ are also "annihilated" or "created" pairwise (here we use the $\mathbb{Z}_2$-equivariance of the short path map and its behavior over the diagonal).

If the parameter t approaches 0 then h_t approaches f^c, because in every $f^{-1}(y)$ we have an even number of points with even multiplicities c_x, so in the expression of h_t we approach every point $x \in f^{-1}(y)$ (such that $c_x = 1$) odd number of times. If t approaches 1 the points $s(x_1, x_2)(t/2)$ and $s(x_2, x_1)(t/2)$ tend to "annihilate" (and do "annihilate" at $t = 1$), and therefore h_1 maps the whole Y to zero cycle. Thus the proof is complete.

Remark 3.1. In [9] a simplified version of the reasoning in [6], in the particular case of the problem of probability of covering by a simplex, was presented, which avoids an explicit use of the space of (co)cycles. In the above proof a similar trick is also possible in the following way: Assume the map f to be generic (or even piece-wise linear). Then for any $y \in Y$ consider the finite set $f^{-1}(y)$ and the full graph (1-dimensional complex) G_y on the vertices $f^{-1}(y)$. Denote the union of these full graphs over $y \in Y$ by G_f. With some natural topology it can be interpreted as an abstract chain (in piece-wise linear case it can be made rigorous) with boundary $\partial G_f = X = \bigcup_{y \in Y} f^{-1}(y)$ (here we use the even degree of f). If any edge of $G_y \subset G_f$ can be realized in X (for example, using a short path map) then G_f is mapped to X continuously. So G_f becomes an $(n+1)$-dimensional chain in $C_{n+1}(X; \mathbb{F}_2)$ with boundary $\partial G_f = [X]$ mod 2. But the fundamental class of X mod 2 cannot vanish, which is a contradiction.

4. Hopf type results

The method of contraction in the space of cycles allows to prove the following generalization of the Hopf theorem:

Theorem 4.1. *Suppose X is a compact manifold of dimension n, Y is an open manifold of dimension n, and $f : X \to Y$ is a continuous map. Assume that X has a metric with*

injectivity radius ρ and $0 < \delta \le \rho$. Then there exist a pair of points $x, y \in X$ at distance δ such that $f(x) = f(y)$.

Remark 4.2. Compared to the Hopf theorem, in this theorem we assume additionally that δ is at most the injectivity radius, but we allow arbitrary open manifold in place of $\mathbb{R}^n$ as the codomain.

Proof. We mostly follow the proof of Theorem 2.3. Assume that f is generic and consider the preimages of a regular value $y \in Y$. Since Y is open the degree of f is even and $f^{-1}(y)$ consists of even number of points.

Assuming that no two points in $f^{-1}(y)$ are at distance δ, make a graph G_y on vertices $f^{-1}(y)$ and edges corresponding to pairs x, y at distance less than δ. By the assumption on the injectivity radius this graph can be drawn by shortest paths on X and depends continuously on y while y does not cross special values of f.

From general position considerations, for any two regular values y' and y'', it is possible to connect them by a path so that the graph G_y while passing from y' to y'' is modified by a series of "simple events" (corresponding to folds of the map f), when a pair of connected vertices x_1 and x_2 is added to G_y and they get connected to the same set of neighbors $N(x_1) \setminus x_2 = N(x_2) \setminus x_1$ in G_y, or a pair of connected vertices x_1 and x_2 with $N(x_1) \setminus x_2 = N(x_2) \setminus x_1$ disappears from G_y.

For generic f, let $S_1 \subset Y$ be the set of special values of f; it has codimension at least 1. Let the set $S_2 \subset S_1$ correspond to "not simple events", that is $S_1 \setminus S_2$ is a fold of f. For generic f, the set S_2 has codimension 2 in Y and its preimage $f^{-1}(S_2)$ has codimension 2 in X. We will ignore S_2 in the reasoning with the fundamental class of X or Y, because the homology is not affected by codimension 2 changes. Now we want to repeat the part of the proof of Theorem 2.3 using the homotopy in the space of cycles along the edges of G_y:

$$h_t(y) = \sum_{(x_1,x_2)\in E(G_y)} s(x_1, x_2)(t/2).$$

Like in Remark 3.1, this homotopy may be interpreted as an $(n+1)$-dimensional chain in X. But unlike the proof of Theorem 2.3, the mod 2 boundary of this chain is *not* the fundamental class $[X]$, but is the set of those points $x \in X \setminus f^{-1}(S_1)$ that come with odd degree in their corresponding graph $G_{f(x)}$. Fortunately, we will show that actually all the points of $X \setminus f^{-1}(S_1)$ have odd degrees in their $G_{f(x)}$ and the proof can be finished similar to the proof of Theorem 2.3.

Without loss of generality assume that X is connected and move a point x' to x'' in $X \setminus f^{-1}(S_2)$ (remember that $f^{-1}(S_2)$ has codimension at least 2 and does not spoil the connectedness). During such a move there may be two possible modifications of the graph $G_{f(x)}$:

1) a pair of vertices (x', x'') disjoint from x is added or removed from $G_{f(x)}$. Since the points x' and x'' have the same sets of neighbors $N(x') \setminus x'' = N(x'') \setminus x'$ then the degree of x is changed by ∓ 2;

2) the vertex x collides with another vertex x' in $G_{f(x)}$ and they exchange. Because their sets of neighbors are the same, $N(x) \setminus x' = N(x') \setminus x$, then the degree of x does not change at this event.

Therefore for any $x \in X \setminus f^{-1}(S_1)$ the degree of x in $G_{f(x)}$ is the same mod 2. Now remember that Y is open and X is closed, then for some $y \in Y$ the graph G_y must be empty and while moving to a nonempty graph it will first generate a pair of points connected by an edge. Hence for some point $x \in X \setminus f^{-1}(S_1)$ its degree in $G_{f(x)}$ must be odd and therefore it must be odd for every $x \in X \setminus f^{-1}(S_1)$. So the image of $h_t(y)$ is a

chain in $C_{n+1}(X;\mathbb{F}_2)$ (see Remark 3.1) with boundary $[X]$ mod 2, which is a contradiction, because X is closed. □

Another approach to Hopf type results is possible, following [16]. Informally, we may increase the dimension of Y, drop the compactness assumption on X, but require an assumption on its Stiefel–Whitney classes:

Theorem 4.3. *Let $f : X \to Y$ be a continuous map between manifolds that induce a zero map on cohomology modulo 2 in positive dimensions. Suppose $\bar{w}_k(TX) \neq 0$ (the dual Stiefel–Whitney class), $\dim X + k - 1 \geq \dim Y$, X is a Riemannian manifold, and δ is a prescribed real number. Then there exists a pair $x, y \in X$ such that $f(x) = f(y)$ and the points x and y are connected by a geodesic of length δ.*

Proof. Consider the space S_X of pairs (x, v), where x is an arbitrary point in X and v is a unit tangent vector at x. This space has a natural $\mathbb{Z}_2$-action $(x, v) \mapsto (x, -v)$.

For $\mathbb{Z}_2$-spaces the following invariant is well-known. The natural $\mathbb{Z}_2$-equivariant map to the one-point space $\pi_{S_X} \to \mathrm{pt}$ induces the map of the equivariant cohomology

$$\pi_{S_X}^* : H_{\mathbb{Z}_2}^*(\mathrm{pt};\mathbb{F}_2) \to H_{\mathbb{Z}_2}^*(S_X;\mathbb{F}_2).$$

The former algebra $H_{\mathbb{Z}_2}^*(\mathrm{pt};\mathbb{F}_2) = H^*(B\mathbb{Z}_2;\mathbb{F}_2) = \mathbb{F}_2[t]$ is a polynomial ring with one-dimensional generator t. The maximal power of t that is mapped nontrivially to the equivariant cohomology of S_X is called the *homological index* of S_X and denoted $\operatorname{ind} S_X$. In [4] the following is proved: take the maximal k so that the dual Stiefel–Whitney class $\bar{w}(TX)$ is nonzero, then

$$\operatorname{ind} S_X = \dim X + k - 1,$$

under the assumption of this theorem $\operatorname{ind} S_X \geq \dim Y$.

Now consider the map $h : S_X \to X$ defined as follows: start a geodesic from x with tangent v and consider its point $h(x, v)$ at distance $\delta/2$ from x. Now the composition $f \circ h$ maps S_X to Y and induces a zero map on the mod 2 cohomology of positive dimension. By the main result from [16] we see that some two pairs (x, v) and $(x, -v)$ should be mapped to the same point, which gives the required pair connected by a geodesic of length δ.

□

5. A possible approach to the multiplicity of generic smooth maps

In the spirit of the proof of Theorem 2.3 it makes sense to consider the following construction. Let X be a closed n-dimensional smooth manifold. Filter the space $cl_0(X;\mathbb{Z})$ (of zero dimensional cycles with integer coefficients) as follows: assign to the cycle $\sum_{x\in X} a_x x$ the norm $\sum_{x\in X} |a_x|$ and denote

$$cl_0^{(k)}(X;\mathbb{Z}) = \{c \in cl_0(X;\mathbb{Z}) : |c| \leq k\}.$$

Consider another n-dimensional closed smooth manifold Y. For a generic smooth map $f : X \to Y$ of degree 0 we construct the cycle map $f^c : Y \to cl_0(X;\mathbb{Z})$ as above. As in the of proof of Theorem 2.3, it is known that the canonical class $\xi \in H^n(cl_0(X;\mathbb{Z});\mathbb{Z})$ does not vanish on $f^c(Y)$.

An important question is estimating from below the *multiplicity* of f (see [6, 8]), which is equal to the least number k such that $cl_0^{(k)}(X;\mathbb{Z}) \supseteq f^c(Y)$. One possible way to show that the multiplicity of any smooth generic $f : X \to Y$ is at least k is to show that the restriction $\xi|_{cl_0^{(k-1)}(X;\mathbb{Z})}$ cannot be detected on a continuous image of Y. A particular case $Y = \mathbb{R}^n \subseteq S^n$ would follow from proving that the restriction $\xi|_{cl_0^{(k-1)}(X;\mathbb{Z})}$ is aspheric (not detectable on $\pi_n(cl_0^{(k-1)}(X;\mathbb{Z}))$).

With the space of cycles $cl_0(X;\mathbb{F}_2)$ with mod 2 coefficients the above construction becomes even more explicit. Since $cl_0(X;\mathbb{F}_2) = \bigcup_{j\geq 0} B(X, 2j)$, we describe the filtering as $cl_0^{(k)}(X;\mathbb{F}_2) = \bigcup_{j=0}^{\lfloor k/2 \rfloor}$. Here the unordered configuration spaces $B(X, 2j)$ resemble the classical approach to the multiplicity using configuration spaces (see [8]).

A remarkable fact is the Dold–Thom–Almgren theorem [1] that describes the homotopy groups of this cycle space:

$$\pi_*(cl_0(X;\mathbb{Z})) = \tilde{H}_*(X;\mathbb{Z}).$$

In [1] it was proved for homology with integer coefficients, but it seems that the version for mod 2 coefficients also works. But it seems that there is no satisfactory description of the homotopy groups of $cl_0^{(k)}(X;\mathbb{Z})$ or $cl_0^{(k)}(X;\mathbb{F}_2)$, which prevents us from developing this approach at the moment.

6. Results on the Urysohn type width

Let us give the definition of the Urysohn type width:

Definition 6.1. Let X be a metric space, Y be arbitrary topological space, denote

$$w(X, Y) = \inf_f \sup_{y\in Y} \operatorname{diam} f^{-1}(y),$$

where inf is taken over all continuous maps $f : X \to Y$.

Informally, the width guarantees that some two far enough points are mapped to a single point under every continuous map $f : X \to Y$. In the book [14] the value $\inf_{\dim Y=k} w(X, Y)$ (the infimum over all k-dimensional polyhedra) is called *the Urysohn width* and denoted $u_k(X)$.

Note that if X is a Riemannian manifold and $\rho(X)$ is its *injectivity radius*, then for every two points $x, y \in X$ at distance $< \rho(X)$ there exists a unique shortest path between x and y depending continuously on the pair (x, y), see [11] for example.

The Borsuk–Ulam theorem asserts that $w(S^n, \mathbb{R}^n) = \pi$ (we measure lengths on the sphere by geodesics), in [16] it is proved that $\mathbb{R}^n$ can be replaced by any open n-dimensional manifold. A general results is:

Corollary 6.2. *The inequality $w(X, Y) \geq \rho(X)$ holds for a compact Riemannian manifold X and an open manifold Y such that $\dim X = \dim Y$.*

Remark 6.3. The particular case $Y = \mathbb{R}^n$ (with $n = \dim X$) of this theorem follows from the Hopf theorem.

Proof. Apply Theorem 2.3 to the shortest path map, which is well defined for pairs $x, y \in X$ closer than $\rho(X)$. The even degree of a map $f : X \to Y$ is guaranteed because Y is open. □

Similarly, Theorem 4.3 implies:

Corollary 6.4. *The inequality $w(X, \mathbb{R}^N) \geq \rho(X)$ holds for a Riemannian manifold X and $N \leq \dim X + k - 1$, where k is the maximum k with $\bar{w}_k(TX) \neq 0$.*

Now we are going to discuss the case when the target space is not a manifold, which is important in the definition of the Urysohn width.

Definition 6.5. Suppose X is a compact Riemannian manifold. Let $\kappa(X)$ be the maximum number such that for any $0 < \delta < \kappa(X)$ any ball in X of radius δ is strictly convex. Call $\kappa(X)$ the *convexity radius.*

Remark 6.6. Obviously $\rho(X) \geq 2\kappa(X)$, because touching strictly convex balls can intersect at one point only. It is also known that $\kappa(X) > 0$ for compact Riemannian manifolds.

Following the reasoning in [14] we prove:

Theorem 6.7. *Let X be a compact n-dimensional Riemannian manifold. Suppose $f : X \to Y$ is a continuous map to a polyhedron. For $\dim X \geq n$, we also require that f is surjective and $f^* : H^n(Y; \mathbb{F}_2) \to H^n(X; \mathbb{F}_2)$ is a zero map. Then for some $y \in Y$*

$$\operatorname{diam} f^{-1}(y) \geq \kappa(X).$$

Before presenting the proof we discuss some consequences of this theorem.

Corollary 6.8. *We have the following estimate for the Urysohn width of an n-dimensional closed Riemannian manifold:*

$$u_{n-1}(X) \geq \kappa(X).$$

Remark 6.9. For a sphere S^n this theorem gives $u_{n-1}(S^n) \geq \pi/2$. In fact, it is known [14, pp. 84–85, 268] that for a unit Euclidean ball $u_{n-1}(B^n) \geq \sqrt{\frac{2n+2}{n}}$ (the edge length of a regular simplex inscribed into B^n), and using a similar method for the half-sphere S^n_+ of the round sphere S^n it can be proved that $u_{n-1}(S^n) \geq u_{n-1}(S^n_+) \geq \arccos\left(-\frac{1}{n}\right)$. Thus Theorem 6.7 is asymptotically sharp.

Remark 6.10. In a private communication E.V. Shchepin told that $u_{n-1}(B^n) = \sqrt{\frac{2n+2}{n}}$. In order to prove the equality we have to give an example of a map. Let us take the orthogonal projection f to the inscribed simplex $\Delta \subseteq B^n$, and then take the PL-map g of the barycentric subdivision of Δ to the cone over the skeleton $0 * \Delta^{n-2}$. The map g is the identity over the skeleton Δ^{n-2}, and maps any center of a facet to 0. It can be checked by hand that any preimage of a point under $g \circ f$ is a cone over a regular simplex of dimension $\leq n$, and its diameter is at most $\sqrt{\frac{2n+2}{n}}$. The values $u_k(B^n)$ are not known precisely for $n/2 < k < n - 1$.

Since every k-dimensional polyhedron can be embedded into some $2k$-dimensional manifold [17, Lemma 7.1], Theorem 2.3 implies:

Corollary 6.11. *We have the following estimate for the Urysohn width of an n-dimensional closed Riemannian manifold for $k \leq \frac{n}{2}$:*

$$u_k(X) \geq \rho(X).$$

Proof of Theorem 6.7. Assume the contrary. Take fine enough subdivision of the polyhedron Y so that for every vertex $v \in Y$ the preimage of $\operatorname{st} v$ has diameter $\leq \delta < \kappa(X)$. This is possible from the standard compactness reasoning.

For every vertex $v \in Y$ consider the set $X_v = f^{-1}(\operatorname{st} v)$, it is nonempty by the assumption. Since $\operatorname{diam} X_v \leq \delta$ we can select a point $\phi(v) \in X_v$ and note that $X_v \subseteq B_\delta(\phi(v))$. Now let us extend the map ϕ from the vertex set of Y to the map $\phi : Y \to X$. If σ is a simplex in Y with vertices $v_1, \dots, v_k$, then the sets $X_{v_1}, \dots, X_{v_k}$ have a common point $p(\sigma)$ (any point in the preimage $f^{-1}(\sigma)$). Therefore the set $\{\phi(v_1), \dots, \phi(v_k)\}$ is contained in $B_\delta(p(\sigma))$ and it makes sense to consider its convex hull $C(\sigma)$ (because $\delta < \kappa(X)$). Note that such convex hulls are homotopy trivial subsets of X, and therefore we can extend ϕ by induction so that for every simplex $\sigma \in Y$ the image $\phi(\sigma)$ is contained in $C(\sigma)$.

Note that the composed map $h = \phi \circ f$ has the following property: $\operatorname{dist}(x, h(x)) \leq \delta$ for any $x \in X$. Indeed, put $y = f(x)$ and let σ be the support simplex of y with vertices

$v_1, \dots, v_k$. We know that $x \in \bigcap_{v\in\sigma} X_v$, therefore $\bigcup_{v\in\sigma} X_v \subseteq B_\delta(x)$. Thus $C(\sigma) \subseteq B_\delta(x)$ and

$$h(x) \in \phi(\sigma) \subseteq C(\sigma) \subseteq B_\delta(x).$$

Since $\delta < \kappa(X) < \rho(x)$ the map h is homotopy equivalent to the identity map of X. Hence $h^* : H^n(X; \mathbb{F}_2) \to H^n(X; \mathbb{F}_2)$ is the identity map, which contradicts the equality $h^* = f^* \circ \phi^*$ and the cohomology condition on f. □

7. Relation to the Gromov waist

In [5] (see also [12]) another notion similar to width was introduced:

Definition 7.1. Let X be a Riemannian manifold of dimension n, Y be a polyhedron of dimension m, denote

$$\gamma(X,Y) = \inf_f \sup_{y\in Y} \operatorname{vol}_{n-m} f^{-1}(y),$$

where inf is taken over all piecewise smooth generic maps $f : X \to Y$. Call $\gamma(X,Y)$ the *waist* of maps $X \to Y$.

It is shown in [5, 12] that $\gamma(S^n, \mathbb{R}^m)$ (here S^n is the round sphere) equals the $(n-m)$-dimensional volume of the equatorial subsphere $S^{n-m} \subset S^n$. In fact, a stronger result in the spirit of concentration phenomena was proved, but we only discuss the result about volumes here.

In [10] is is shown that $\mathbb{R}^m$ on the right hand side can be replaced with arbitrary m-dimensional manifold. The open question (see [5, Question of page 194]) is:

Question 7.2. Can one one bound from below $\gamma(S^n, K)$ (defined as $(n-m)$-dimensional volume) for any m-dimensional polyhedron K?

In this section we investigate this question and some its generalizations in the simplest case, when the map f has codimension -1 (i.e. drops dimension by 1). In this case the preimage of a point is generally something 1-dimensional.

Theorem 7.3. *Let Y be an $(n-1)$-dimensional polyhedron, then $\gamma(S^n, Y) \geq \pi$.*

Proof. Let $f : S^n \to Y$ be a generic smooth map. Put $X_y = \pi_0(f^{-1}(y))$, that is the set of all connected components of $f^{-1}(y)$. On the union

$$\tilde{Y} = \bigcup_{y\in Y} X_y$$

we define the metric as the minimal distance between disjoint compacta. The map f is a composition of continuous maps $f = g \circ \tilde{f}$ so that

$$S^n \xrightarrow{\tilde{f}} \tilde{Y} \xrightarrow{g} Y.$$

Note that all the preimages $g^{-1}(y)$ are zero-dimensional, so $\tilde{Y}$ has dimension at most $n-1$.

Using the result of K. Sitnikov, cited in [14, p. 268], we note that for some point $\tilde{y} \in \tilde{Y}$ the preimage $W = \tilde{f}^{-1}(\tilde{y})$ is not contained in an open half-sphere of S^n. By the definition of $\tilde{Y}$, the set W is connected, hence it intersects any equatorial subsphere $S^{n-1} \subset S^n$. By subdividing W into smaller segments it is easy to show that the probability of intersection between W and some equatorial S^{n-1} is at most $\dfrac{\operatorname{vol}_1 W}{\pi}$, so $\operatorname{vol}_1 W \geq \pi$. □

Remark 7.4. Note that the estimate of [5, 12] is better: $\operatorname{vol}_1 W \geq 2\pi$. This can also be explained in the above terms: if Y is a manifold, then for a generic f the set W is a manifold, i.e. topologically a circle. Hence it intersects a generic equatorial subsphere in at least two points!

Now it makes sense to state the following conjecture, which would answer Question 7.2:

Conjecture 7.5. *Let Y be an m-dimensional polyhedron, $n > m$. For every generic map $f : S^n \to Y$ there is a connected subset $W \subseteq S^n$ such that $f(W)$ is a point and W intersects any equatorial subsphere $S^m \subset S^n$.*

We also extend Theorem 7.3 to any Riemannian manifold as the domain space:

Theorem 7.6. *Let X be a compact n-dimensional Riemannian manifold and Y be an $(n-1)$-dimensional polyhedron, then $\gamma(X, Y) \geq 2\kappa(X)$.*

Proof. First pass to the map $\tilde{f} : X \to \tilde{Y}$ with connected preimages, as in the proof of Theorem 7.3.

Now assume the contrary: for any $\tilde{y} \in \tilde{Y}$ the preimage $W = \tilde{f}^{-1}(\tilde{y})$ has length $< 2\kappa$. From the compactness reasoning we may assume that $\operatorname{vol}_1 W < 2\delta < 2\kappa$ for all W. Since W is connected it has a unique ball $B(W)$ of radius $\leq \delta$ that contains W and is the least ball containing W. The rest of the proof follows the proof of Theorem 6.7 by constructing a map $h : X \to X$, which has $(n-1)$-dimensional image and is homotopic to the identity because of the condition $\operatorname{dist}(h(x), x) \leq \delta$. □

We can improve Theorem 7.6 in the case, when Y is a manifold, thus extending the main result of [5, 12, 10] (in the particular case of codimension -1 maps) to any $\mathbf{CAT}(1)$ metric space as the domain:

Theorem 7.7. *Let X be a compact n-dimensional manifold with $\mathbf{CAT}(1)$ metric and Y be an $(n-1)$-dimensional manifold, then $\gamma(X, Y) \geq 2\pi$.*

The proof will follow from several lemmas, all the spheres are assumed to be the standard Euclidean unit spheres.

Lemma 7.8. *Suppose C is a closed path on the unit sphere S^2 with length not greater than 2π. Then C lies in a hemisphere.*

Proof. We need the Kirszbraun theorem for the sphere [15]. It asserts that for any $U \subset S^2$ and any 1-Lipschitz map $f : U \to S^2$ there exists 1-Lipschitz extension $\overline{f} : S^2 \to S^2$ (i.e $\overline{f}|_U = f$).

Now consider a geodesic circle σ on S^2 and let z be one of its two centers. There obviously exists a 1-Lipschitz map $f : \sigma \to C$. This map can be extended to a 1-Lipschitz map $\overline{f} : S^2 \to S^2$. Thus the distance from the point $\overline{f}(z)$ to any point in $C = f(\sigma)$ is at most $\pi/2$. This means that the hemisphere with center $\overline{f}(x)$ contains C. □

Lemma 7.9. *Let $[am]$ be the median of the spherical triangle (possibly degenerate) $\triangle abc \subset S^2$, and $d(a, b) + d(a, c) < \pi$. Then*

$$d(a, m) \leq \frac{1}{2}(d(a, b) + d(a, c)).$$

Proof. Suppose a' is a reflection of a with respect to m. The perimeter of the quadrangle $aba'c$ is less than 2π. By Lemma 7.8 the quadrangle $aba'c$ is contained in a hemisphere, and therefore $d(a, a') = d(a, m) + d(m, a') = 2d(a, m)$.

Applying the triangle inequality to $\triangle aa'b$ we obtain:

$$2d(a, m) = d(a, a') \leq d(a, b) + d(b, a') = d(a, b) + d(a, c).$$

$\square$

Lemma 7.10. *Let C be a closed curve in X (a* $\mathbf{CAT}(1)$ *metric space) with length $\ell(C) < 2\pi$. Then C can be covered by a ball with radius $\leq \ell(C)/4 < \pi/2$.*

Proof. Denote $\ell(C)$ simply by ℓ. Let a and b be two points on C that correspond to parameters 0 and $\ell/2$. Suppose m is a midpoint of the geodesic $[ab]$. Let us show that the ball with center at m and radius $\ell/4$ covers the whole C.

Let x be any point on C with parameter α. Without loss of generality we assume that $\alpha < \ell/2$. Then $d(x, a) \leq \alpha$ and $d(x, b) \leq \ell/2 - \alpha$.

By the definition of a $\mathbf{CAT}(1)$ metric space, the distance $d(m, x)$ is less or equal to the corresponding median in the model triangle on S^2. In other words, if we consider a triangle $\triangle a'b'x'$ with the same as $\triangle abx$ side lengths on S^2; this is possible because the perimeter of $\triangle abx$ is $< 2\pi$. Then the length of the median $[x'm']$ will be $\geq d(x, m)$. Applying lemma 7.9 we obtain: $d(x', m') \leq \ell/4$, and therefore $d(x, m) \leq \ell/4$. This is exactly what we need. $\square$

Lemma 7.11. *Any ball of radius $< \pi/2$ in a* $\mathbf{CAT}(1)$ *metric space X is strictly convex, i.e. $\kappa(X) \geq \pi/2$.*

See [2, Proposition 9.1.16] for the proof.

Proof of Theorem 7.7. Note that for any generic $f : X \to Y$ and a generic $y \in Y$ the preimage $f^{-1}(y)$ is a union of closed curves. Then we argue as in the proof of Theorem 7.6 and use Lemmas 7.10 and 7.11. $\square$

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